Trouvez les ACF du processus suivant : $y_t = \phi_1 y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$ avec $\varepsilon_t \sim i.i.d.(0, \sigma_\varepsilon^2)$

$$\gamma_0 = \operatorname{cov}(y_t, y_t) = \operatorname{var}(y_t) = ?$$

First to obtain the $\gamma_0 = \operatorname{var}(y_t)$ we need to use the following formula:

$$\gamma_0 = \operatorname{var}(y_t) = E(y_t y_t) - E(y_t)E(y_t) = E(y_t^2) - E(y_t)E(y_t)$$

If the process is stationary

$$\underbrace{E(y_t)}_{\mu_y} = \phi_1 \underbrace{E(y_{t-1})}_{\mu_y} + \underbrace{E(\varepsilon_t)}_{0} + \theta \underbrace{E(\varepsilon_{t-1})}_{0}$$

So

$$\mu_y = \phi_1 \mu_y + 0 \quad \Rightarrow \quad \mu_y - \phi_1 \mu_y = 0 \quad \Rightarrow \quad \mu_y = 0 / (1 - \phi_1) = 0$$

Since there is no constant in this model, the mean of the process is 0. Also notice that if this process is stationary it admits an MA(?) representation with no constant, and in that case $E(y_t) = E(MA(?)) = 0$

Thus

$$\gamma_0 = \operatorname{var}(y_t) = E(y_t y_t) - \underbrace{E(y_t)}_{0}\underbrace{E(y_t)}_{0} = E(y_t y_t)$$

To calculate $\gamma_0 = \operatorname{var}(y_t) = E(y_t y_t)$ let us take the process $y_t = \phi_1 y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$ and pre-multiply it by y_t and then take the expectation

$$y_t y_t = \phi_1 y_t y_{t-1} + y_t \varepsilon_t + \theta_1 y_t \varepsilon_{t-1}$$

Taking the expectation

$$\gamma_0 = \operatorname{var}(y_t) = E(y_t y_t) = \phi_1 \underbrace{E(y_t y_{t-1})}_{\gamma_1} + E(y_t \varepsilon_t) + \theta_1 E(y_t \varepsilon_{t-1})$$

(*)

with

$$E(y_t \varepsilon_t) = E(y_t \varepsilon_t) = E([\phi_1 y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}]\varepsilon_t) = \phi_1 \underbrace{E(y_{t-1}\varepsilon_t)}_{0} + \underbrace{E(\varepsilon_t \varepsilon_t)}_{\sigma_\varepsilon^2} + \theta_1 \underbrace{E(\varepsilon_{t-1}\varepsilon_t)}_{0} = \sigma_\varepsilon^2$$

and

$$E(y_t \varepsilon_{t-1}) = E[(\phi_1 y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1})\varepsilon_{t-1}] = \phi_1 \underbrace{E[y_{t-1}\varepsilon_{t-1}]}_{\substack{\sigma_\varepsilon^2 \text{ from the} \\ \text{above equation} \\ \text{lagged once}}} + \underbrace{E[\varepsilon_t \varepsilon_{t-1}]}_{0} + \theta_1 \underbrace{E[\varepsilon_{t-1}\varepsilon_{t-1}]}_{\sigma_\varepsilon^2} = \phi_1 \sigma_\varepsilon^2 + \theta_1 \sigma_\varepsilon^2$$

Plugging all this into (*)

$$\gamma_0 = \operatorname{var}(y_t) = E(y_t y_t) = \phi_1 \underbrace{E(y_t y_{t-1})}_{\gamma_1} + E(y_t \varepsilon_t) + \theta_1 E(y_t \varepsilon_{t-1}) = \phi_1 \gamma_1 + \sigma_\varepsilon^2 + \theta_1 (\phi_1 \sigma_\varepsilon^2 + \theta_1 \sigma_\varepsilon^2) \qquad (**)$$

Now for $E(y_t y_{t-1})$

We multiply $y_t = \phi_1 y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$ by y_{t-1} which gives us

$$y_{t-1}y_t = \phi_1 y_{t-1}y_{t-1} + y_{t-1}\varepsilon_t + \theta_1 y_{t-1}\varepsilon_{t-1}$$

Then we take expectations

$$\gamma_1 = \underbrace{E(y_{t-1}y_t)}_{\gamma_1} = \phi_1 \underbrace{E(y_{t-1}y_{t-1})}_{\gamma_0} + \underbrace{E(y_{t-1}\varepsilon_t)}_{0} + \theta_1 \underbrace{E(y_{t-1}\varepsilon_{t-1})}_{\sigma_\varepsilon^2} = \phi_1 \gamma_0 + \theta_1 \sigma_\varepsilon^2 \qquad (***)$$

Plugging this into (**)

We get

$$\gamma_0 = \phi_1 \gamma_1 + \sigma_\varepsilon^2 + \theta_1 (\phi_1 \sigma_\varepsilon^2 + \theta_1 \sigma_\varepsilon^2) = \phi_1 (\phi_1 \gamma_0 + \theta_1 \sigma_\varepsilon^2) + \sigma_\varepsilon^2 + \theta_1 (\phi_1 \sigma_\varepsilon^2 + \theta_1 \sigma_\varepsilon^2)$$

And thus we get

$$\gamma_0 - \phi_1^2 \gamma_0 = \phi_1 \theta_1 \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + \theta_1 (\phi_1 \sigma_\varepsilon^2 + \theta_1 \sigma_\varepsilon^2)$$

$$\gamma_0 = \frac{\phi_1 \theta_1 \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + \theta_1 (\phi_1 \sigma_\varepsilon^2 + \theta_1 \sigma_\varepsilon^2)}{1 - \phi_1^2} = \sigma_\varepsilon^2 \left(\frac{\phi_1 \theta_1 + 1 + \theta_1 (\phi_1 + \theta_1)}{1 - \phi_1^2} \right) = \sigma_\varepsilon^2 \left(\frac{\phi_1 \theta_1 + 1 + \theta_1 \phi_1 + \theta_1^2}{1 - \phi_1^2} \right)$$

$$\gamma_0 = \sigma_\varepsilon^2 \left(\frac{1 + 2\theta_1 \phi_1 + \theta_1^2}{1 - \phi_1^2} \right)$$

We already found out γ_1 given by equation (***)

Then for j=2

$$\gamma_2 = \underbrace{E(y_{t-2}y_t)}_{\gamma_2} = \phi_1 \underbrace{E(y_{t-2}y_{t-1})}_{\gamma_1} + \underbrace{E(y_{t-2}\varepsilon_t)}_{0} + \theta_1 \underbrace{E(y_{t-2}\varepsilon_{t-1})}_{0} = \phi_1 \gamma_1$$

We can see that for $\forall j \geq 2$ we have a recurrence such that $\gamma_j = \phi_1 \gamma_{j-1} = \phi_1^{j-1} \gamma_1$

So we have the following system of equations:

$$\gamma_0 = \phi_1 \gamma_1 + \sigma_\varepsilon^2 + \theta_1 (\phi_1 \sigma_\varepsilon^2 + \theta_1 \sigma_\varepsilon^2) = \sigma_\varepsilon^2 \left(\frac{1 + 2\theta_1 \phi_1 + \theta_1^2}{1 - \phi_1^2} \right)$$

$$\gamma_1 = \underbrace{E(y_{t-1}y_t)}_{\gamma_1} = \phi_1 \gamma_0 + \theta_1 \sigma_\varepsilon^2$$

$$\gamma_j = \phi_1 \gamma_{j-1} = \phi_1^{j-1} \gamma_1 \quad \forall j \geq 2$$

Now for the ACF we have:

$$\rho_0 = \frac{\gamma_0}{\gamma_0} = 1$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\phi_1 \gamma_0 + \theta_1 \sigma_\varepsilon^2}{\gamma_0} = \frac{\phi_1 \sigma_\varepsilon^2 \left(\dfrac{1 + 2\theta_1 \phi_1 + \theta_1^2}{1 - \phi_1^2} \right) + \theta_1 \sigma_\varepsilon^2}{\sigma_\varepsilon^2 \left(\dfrac{1 + 2\theta_1 \phi_1 + \theta_1^2}{1 - \phi_1^2} \right)}$$

$$= \frac{\sigma_\varepsilon^2 \left(\dfrac{\phi_1 + 2\theta_1 \phi_1^2 + \phi_1 \theta_1^2}{1 - \phi_1^2} + \dfrac{(1 - \phi_1^2)}{(1 - \phi_1^2)} \theta_1 \right)}{\sigma_\varepsilon^2 \left(\dfrac{1 + 2\theta_1 \phi_1 + \theta_1^2}{1 - \phi_1^2} \right)} = \frac{\phi_1 + 2\theta_1 \phi_1^2 + \phi_1 \theta_1^2 + (1 - \phi_1^2)\theta_1}{1 + 2\theta_1 \phi_1 + \theta_1^2}$$

$$= \frac{\phi_1 + 2\theta_1 \phi_1^2 + \phi_1 \theta_1^2 + \theta_1 - \theta_1 \phi_1^2}{1 + 2\theta_1 \phi_1 + \theta_1^2} = \frac{\phi_1 + \theta_1 \phi_1^2 + \phi_1 \theta_1^2 + \theta_1}{1 + 2\theta_1 \phi_1 + \theta_1^2}$$

$$= \frac{\phi_1 + \theta_1 \phi_1^2 + \phi_1 \theta_1^2 + \theta_1}{1 + 2\theta_1 \phi_1 + \theta_1^2} = \frac{\phi_1 (1 + \theta_1 \phi_1) + \phi_1 \theta_1^2 + \theta_1}{1 + 2\theta_1 \phi_1 + \theta_1^2} = \frac{\phi_1 (1 + \theta_1 \phi_1) + \theta_1 (\phi_1 \theta_1 + 1)}{1 + 2\theta_1 \phi_1 + \theta_1^2}$$

$$= \frac{(\phi_1 + \theta_1)(1 + \theta_1 \phi_1)}{1 + 2\theta_1 \phi_1 + \theta_1^2} = \rho_1$$

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \frac{\phi_1 \gamma_{j-1}}{\gamma_0} = \phi_1 \rho_{j-1} = \frac{\phi_1^{j-1} \gamma_1}{\gamma_0} = \phi_1^{j-1} \rho_1 \quad \forall j \geq 2$$

That`s it!

For ARMA(p,q)

We can write a finite order ARMA(p,q) process such that

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + ... + +\theta_q \varepsilon_{t-q}$$

Or

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - .... - \phi_p y_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + ... + +\theta_q \varepsilon_{t-q}$$

We can also write the same ARMA(p,q) in the following polynomial representation

$$\phi(L) y_t = \theta(L) \varepsilon_t$$

Where $\phi(z) = 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p$ and where $\theta(z) = 1 + \theta_1 L + \theta_2 L^2 + ... + \theta_q L^q$

Stationary process

For an ARMA(p,q) process to be **stationary** (so that we can rewrite it as a pure MA(?) representation), all the roots of the characteristic polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - ... - \phi_p z^p = 0$ corresponding to the AR part must lie outside the unit circle.

This means that for complex roots of the form $C = A + Bi$ where $i = \sqrt{-1}$, all the roots z_i must be such that $\sqrt{A^2 + B^2} > 1$.

For **real roots,** that simply means that $\sqrt{A^2 + 0} = |A| > 1$ which is simply the restriction that $|z_i| > 1$

Invertible process

For an ARMA(p,q) process to be invertible (so that we can rewrite it as a pure AR(?) representation), all the roots of the characteristic polynomial $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + ... + \theta_q z^q = 0$ corresponding to the MA part must lie outside the unit circle.

This means that for complex roots of the form $C = A + Bi$ where $i = \sqrt{-1}$, all the roots z_i must be such that $\sqrt{A^2 + B^2} > 1$.

For **real roots,** that simply means that $\sqrt{A^2 + 0} = |A| > 1$ which is simply the restriction that $|z_i| > 1$