Trouvez les ACF et les PACF pour le processus AR(2) suivant : $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$ avec $\varepsilon_t \sim i.i.d(0,\sigma_\varepsilon^2)$

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First to obtain the $\gamma_0 = \text{var}(y_t)$

we need to use the following formula :

$$\gamma_0 = \text{var}(y_t) = E(y_t\, y_t) - E(y_t)E(y_t) = E(y_t^2) - E(y_t)E(y_t)$$

If the process is stationary

$$\underbrace{E(y_t)}_{\mu_y} = \phi_1 \underbrace{E(y_{t-1})}_{\mu_y} + \phi_2 \underbrace{E(y_{t-2})}_{\mu_y} + \underbrace{E(\varepsilon_t)}_{0}$$

So

$$\mu_y = \phi_1\mu_y + \phi_2\mu_y + 0 \quad \Rightarrow \quad \mu_y - \phi_1\mu_y - \phi_2\mu_y = 0 \quad \Rightarrow \quad \mu_y = 0/(1-\phi_1-\phi_2) = 0$$

Since there is no constant in this model, the mean of the process is 0. Also notice that if this process is stationary it admits an MA(?) representation with no constant, and in that case $E(y_t) = E(MA(?)) = 0$

Thus

$$\gamma_0 = \text{var}(y_t) = E(y_t\, y_t) - \underbrace{E(y_t)}_{0}\underbrace{E(y_t)}_{0} = E(y_t\, y_t)$$

To calculate $\gamma_0 = \text{var}(y_t) = E(y_t\, y_t)$

Let us take the process $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$ and pre-multiply it by y_t and then take the expectation

$$y_t y_t = \phi_1 y_t y_{t-1} + \phi_2 y_t y_{t-2} + y_t\varepsilon_t$$

Taking the expectation

$$\gamma_0 = \text{var}(y_t) = E(y_t y_t) = \phi_1 E(y_t y_{t-1}) + \phi_2 E(y_t y_{t-2}) + E(y_t\varepsilon_t)$$

Now using the definition

$$\gamma_0 = \text{var}(y_t) = E(y_t y_t) = \phi_1 \underbrace{E(y_t y_{t-1})}_{\gamma_{-1}} + \phi_2 \underbrace{E(y_t y_{t-2})}_{\gamma_{-2}} + E(y_t\varepsilon_t)$$

$$E(y_t\varepsilon_t) = E[(\phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t)\varepsilon_t] = \phi_1 \underbrace{E[y_{t-1}\varepsilon_t]}_{0} + \phi_2 \underbrace{E[y_{t-2}\varepsilon_t]}_{0} + \underbrace{E[\varepsilon_t\varepsilon_t]}_{\sigma_\varepsilon^2} = \sigma_\varepsilon^2$$

$E[y_{t-1}\varepsilon_t] = 0$ because $\varepsilon_t \sim i.i.d(0,\sigma_\varepsilon^2)$ and it is therefore uncorrelated with the passed value y_{t-1}

$E[y_{t-2}\varepsilon_t] = 0$ because $\varepsilon_t \sim i.i.d(0,\sigma_\varepsilon^2)$ and it is therefore uncorrelated with the passed value y_{t-2}

So

$$\gamma_0 = \text{var}(y_t) = E(y_t y_t) = \phi_1\gamma_{-1} + \phi_2\gamma_{-2} + \sigma_\varepsilon^2 = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma_\varepsilon^2$$

Since the autocovariances are symmetric $\gamma_{-j} = \gamma_j$

Now for the $\gamma_1 = \text{cov}(y_t, y_{t-1}) = E(y_t y_{t-1}) = E(y_{t-1} y_t)$

Let us pre-multiply y_t by y_{t-1} and then take the expectation

$$y_{t-1} y_t = y_{t-1}(\phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t) = \phi_1 y_{t-1} y_{t-1} + \phi_2 y_{t-1} y_{t-2} + y_{t-1}\varepsilon_t$$

$$\gamma_1 = \text{cov}(y_t, y_{t-1}) = E(y_{t-1}y_t) = E[\phi_1 y_{t-1}y_{t-1} + \phi_2 y_{t-1}y_{t-2} + y_{t-1}\varepsilon_t]$$

$$= \phi_1 \underbrace{E[y_{t-1}y_{t-1}]}_{\gamma_0} + \phi_2 \underbrace{E[y_{t-1}y_{t-2}]}_{\gamma_{-1}} + \underbrace{E[y_{t-1}\varepsilon_t]}_{0} = \phi_1\gamma_0 + \phi_2\gamma_{-1} = \phi_1\gamma_0 + \phi_2\gamma_1$$

So $\gamma_1 = \phi_1\gamma_{1-1} + \phi_2\gamma_{1-2} = \phi_1\gamma_0 + \phi_2\gamma_{-1} = \phi_1\gamma_0 + \phi_2\gamma_1$

so $\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1 \quad \Rightarrow \quad \gamma_1 - \phi_2\gamma_1 = \phi_1\gamma_0 \quad \Rightarrow \quad \gamma_1 = \dfrac{\phi_1\gamma_0}{1-\phi_2}$

Now for the $\gamma_2 = \text{cov}(y_t, y_{t-2}) = E(y_t y_{t-2}) = E(y_{t-2}y_t)$
Let us pre-multiply y_t by y_{t-2} and then take the expectation

$$y_{t-2}y_t = y_{t-2}(\phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t) = \phi_1 y_{t-2}y_{t-1} + \phi_2 y_{t-2}y_{t-2} + y_{t-2}\varepsilon_t$$

$$\gamma_2 = \text{cov}(y_t, y_{t-2}) = E(y_{t-2}y_t) = E[\phi_1 y_{t-2}y_{t-1} + \phi_2 y_{t-2}y_{t-2} + y_{t-2}\varepsilon_t]$$

$$= \phi_1 \underbrace{E[y_{t-2}y_{t-1}]}_{\gamma_1} + \phi_2 \underbrace{E[y_{t-2}y_{t-2}]}_{\gamma_0} + \underbrace{E[y_{t-2}\varepsilon_t]}_{0} = \phi_1\gamma_1 + \phi_2\gamma_0$$

replacing $\gamma_1 = \dfrac{\phi_1\gamma_0}{1-\phi_2}$ into the above expression we get:

$$\gamma_2 = \phi_1\gamma_{2-1} + \phi_2\gamma_{2-2} = \phi_1\gamma_1 + \phi_2\gamma_0 = \phi_1\left(\frac{\phi_1\gamma_0}{1-\phi_2}\right) + \phi_2\gamma_0 = \left(\frac{\phi_1^2\gamma_0}{1-\phi_2}\right) + \phi_2\gamma_0$$

Continuing to γ_j for j=3,4,… one can see that there is a recurrence which turns out to be

$$\gamma_j = \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2} \qquad \forall j = 1,2,3,...$$

To sum up all auto-covariances are given by

$\gamma_0 = \phi_1\gamma_{-1} + \phi_2\gamma_{-2} + \sigma_\varepsilon^2$

$\gamma_1 = \phi_1\gamma_{1-1} + \phi_2\gamma_{1-2} = \phi_1\gamma_0 + \phi_2\gamma_{-1} = \phi_1\gamma_0 + \phi_2\gamma_1$

$\gamma_2 = \phi_1\gamma_1 + \phi_2\gamma_0 = \phi_1\left(\dfrac{\phi_1\gamma_0}{1-\phi_2}\right) + \phi_2\gamma_0 = \left(\dfrac{\phi_1^2\gamma_0}{1-\phi_2}\right) + \phi_2\gamma_0$

$\gamma_j = \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2}$ for $j = 3,...$

This is a system of equations

We could write this system compactly as a system of 3=p+1 equations (for AR(p) where p=2)

$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma_\varepsilon^2$

$\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1$

$\gamma_2 = \phi_1\gamma_1 + \phi_2\gamma_0$

And with this we could solve for $\gamma_0, \gamma_1, \gamma_2$ given the parameters of the process ϕ_1, ϕ_2.
This is all that is needed to get the rest of the γ_j for j=3,4,… because recursively $\gamma_j = \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2}$

To obtain the ACF, the autocorrelation function we need to calculate $\rho_j = \dfrac{\gamma_j}{\gamma_0} \quad \forall j$

$$\rho_0 = \frac{\gamma_0}{\gamma_0} = \frac{\phi_1\gamma_1 + \phi_2\gamma_2 + \sigma_\varepsilon^2}{\phi_1\gamma_1 + \phi_2\gamma_2 + \sigma_\varepsilon^2} = 1$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\phi_1\gamma_0 + \phi_2\gamma_{-1}}{\gamma_0} = \frac{\phi_1\gamma_0 + \phi_2\gamma_1}{\gamma_0} = \phi_1\rho_0 + \phi_2\rho_1 = \frac{\phi_1\gamma_0 + \phi_2\left(\dfrac{\phi_1\gamma_0}{1-\phi_2}\right)}{\gamma_0} = \phi_1 + \phi_2\left(\frac{\phi_1}{1-\phi_2}\right) = \phi_1\gamma_0 + \phi_2\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_{-1}$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\phi_1\gamma_1 + \phi_2\gamma_0}{\gamma_0} = \phi_1\frac{\gamma_1}{\gamma_0} + \phi_2\frac{\gamma_0}{\gamma_0} = \phi_1\rho_1 + \phi_2\rho_0 = \frac{\left(\dfrac{\phi_1^2\gamma_0}{1-\phi_2}\right) + \phi_2\gamma_0}{\gamma_0} = \left(\frac{\phi_1^2}{1-\phi_2}\right) + \phi_2$$

$\rho_j = \dfrac{\gamma_j}{\gamma_0} = \phi_1\dfrac{\gamma_{j-1}}{\gamma_0} + \phi_2\dfrac{\gamma_{j-2}}{\gamma_0} = \phi_1\rho_{j-1} + \phi_2\rho_{j-2}$ for $j = 3,...$ Using the fact that $\gamma_j = \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2}$ for $j = 3,...$

So the look of the ACF for an AR(2) will decay in some steady fashion (which might be sinusoidal). The autocorrelations eventually die out geometrically.

Bingo!!!

This is a system of equations referred to as the Yule-Walker equations (or Luke Skywalker equations)

We could write this system compactly as a system of 3=p+1 equations (for AR(p) where p=2)

$$\rho_0 = 1$$
$$\rho_1 = \phi_1\rho_0 + \phi_2\rho_1 = \phi_1 + \phi_2\rho_1$$
$$\rho_2 = \phi_1\rho_1 + \phi_2\rho_0 = \phi_1\rho_1 + \phi_2$$

This is all that is needed to get the rest of the ρ_j for j=3,4,… because recursively $\rho_j = \phi_1\rho_{j-1} + \phi_2\rho_{j-2}$

Using the equations for ρ_1, ρ_2 we can write the system of 2 unknown and 2 equations

$$\underbrace{\begin{bmatrix}\rho_1 \\ \rho_2\end{bmatrix}_{2\times1}}_{\boldsymbol{\rho}} = \underbrace{\begin{bmatrix}1 & \rho_1 \\ \rho_1 & 1\end{bmatrix}_{2\times2}}_{\mathbf{P}} \underbrace{\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix}_{2\times1}}_{\boldsymbol{\Phi}} \qquad \text{or} \qquad \boldsymbol{\rho} = \mathbf{P}\boldsymbol{\Phi}$$

So the solution would be $\boldsymbol{\Phi} = \mathbf{P}^{-1}\boldsymbol{\rho}$ if $\mathbf{P}$ is definite positive.

So with this we can solve for the parameters of the process ϕ_1, ϕ_2 given ρ_1, ρ_2.

Now for the **PACF, the partial autocorrelation function**

We have that $\rho_j = \phi_1\rho_{j-1} + \phi_2\rho_{j-2} + ... + \phi_j\rho_{j-p} = \sum_{h=1}^{p}\phi_h\rho_{j-h}$ for all j>0

We also defined the partial autocorrelation of order j, denoted ϕ_{jj}

By the following regression equation $y_t = \phi_{j1}y_{t-1} + \phi_{j2}y_{t-2} + ... + \phi_{jj}y_{t-j} + \varepsilon_t = \sum_{h=1}^{j}\phi_{jh}y_{t-h} + \varepsilon_t$

Remember that we can solve the system given by the Yule-Walker equations

$\rho_j = \phi_1\rho_{j-1} + \phi_2\rho_{j-2} + ... + \phi_j\rho_{j-p} = \sum_{h=1}^{j}\phi_h\rho_{j-h}$ for all j=1,2,3,…,J

Supposing each time that the process is of order j=p, we could write using the definition of $\phi_{jj} = \phi_j$ when j=p

$\rho_j = \phi_{j1}\rho_{j-1} + \phi_{j2}\rho_{j-2} + ... + \phi_{jj}\rho_{j-p} = \sum_{h=1}^{j}\phi_{jh}\rho_{j-h}$ for all j=1,2,3,…,J

Which gives us the different ϕ_{jj} when h=j for all j=1,2,3,…,J

Let us write the system in extensive form, to see more clearly what's going on for a AR(2) process.

Supposing that the process is such that p=1, we would get

$\rho_1 = \phi_1\rho_0 = \phi_{11}\rho_0$

$\rho_1 = \phi_{11}\rho_0 = \phi_{11}$

So $\phi_{11} = \rho_1$

For an AR(2) supposing that p=2

$\rho_1 = \phi_1\rho_{1-1} + \phi_2\rho_{1-2} = \phi_1\rho_0 + \phi_2\rho_{-1} = \phi_1\rho_0 + \phi_2\rho_1$ and supposing that the process is such that p=2, we would get

$\rho_1 = \phi_{21}\rho_0 + \phi_{22}\rho_1 = \phi_{21} + \phi_{22}\rho_1$ (1)

$\rho_2 = \phi_1\rho_1 + \phi_2\rho_0 = \phi_1\rho_1 + \phi_{22}\rho_0$ and supposing that the process is such that p=2, we would get

$\rho_2 = \phi_{21}\rho_1 + \phi_{22}$ (2)

Solving for ϕ_{11} and ϕ_{22} we will use equation (1) and (2)

From (2) we can write

$$\phi_{21} = \frac{\rho_2 - \phi_{22}}{\rho_1}$$

Substituting this into (1)

We get $\rho_1 = \phi_{21} + \phi_{22}\rho_1 = \left(\frac{\rho_2 - \phi_{22}}{\rho_1}\right) + \phi_{22}\rho_1$

Now solving for ϕ_{22} by multiply in the above expression by ρ_1

$\rho_1^2 = \rho_2 - \phi_{22} + \phi_{22}\rho_1^2 = \rho_2 + (\rho_1^2 - 1)\phi_{22}$

$$\rho_1^2 - \rho_2 = (\rho_1^2 - 1)\phi_{22}$$

$$\phi_{22} = \frac{\rho_1^2 - \rho_2}{\rho_1^2 - 1} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

And we could go on forever for all ϕ_{jj}

But it turns out that for all ϕ_{jj} where j>2 for an AR(2) we have that $\phi_{jj} = 0$ by pure logic, since the parameters $\phi_j = 0 \ \forall j = 3, 4, ...$ that is for $\forall j > p = 2$

Notice that the PACF cuts after 2